Asynchronous Circuit Design

Chris J. Myers
Lecture B: Sets and Relations
Appendix B

Sets

- A set \( S \) is any collection of objects.
- Each \( x \) in \( S \) is a member of \( S \) (denoted \( x \in S \)).
- When \( x \) is not a member of \( S \), it is denoted by \( x \notin S \).
- Two sets \( X \) and \( Y \) are equal when they consist of the same members (denoted \( X = Y \)).
- This means that if \( X = Y \) and \( a \in X \), then \( a \in Y \).
- This is known as the principle of extension.
- If two sets are not equal, it is denoted \( X \neq Y \).
- There are three basic properties of equality:
  - \( X = X \) (reflexive)
  - \( X = Y \) implies \( Y = X \) (symmetric)
  - \( X = Y \) and \( Y = Z \) then \( X = Z \) (transitive)

Principle of Abstraction

Large or infinite sets are described using the help of predicates.
- A predicate \( P(x) \) takes an object and returns true or false.
- When a set \( S \) is defined using a predicate \( P(x) \), the set \( S \) contains those objects a such that \( P(a) \) is true.
- This is known as the principle of abstraction.
- This is denoted using set builder notation as follows:
  \[ S = \{ x \mid P(x) \} \]
- Read as “the set of all objects \( x \) such that \( P(x) \) is true.”
- The following sets can be used interchangeably:
  \[ \{ x \mid x \in A \text{ and } P(x) \} = \{ x \in A \mid P(x) \} \]
  \[ \{ y \mid y = f(x) \text{ and } P(y) \} = \{ f(x) \mid P(x) \} \]

Examples

\{ x \in \mathbb{N} \mid x \text{ divides } 30 \} =

Which of the following sets are equal to it?
\{ 30, 15, 10, 6, 5, 3, 2, 1 \}
\{ 1, 2, 3, 5, 6, 10, 15, 30 \}
\{ 1, 2, 3, 4, 5, 6, 10, 15, 30 \}
\{ 1, 2, 3, 5, 6, 10, 15, 30 \}

Subset

- If \( X \) and \( Y \) are sets such that every member of \( X \) is also a member of \( Y \),
  then \( X \) is a subset of \( Y \) (denoted \( X \subseteq Y \)).
- If every member of \( Y \) is a member of \( X \), then \( X \) is a superset of \( Y \)
  (denoted \( X \supseteq Y \)).
- If \( X \subseteq Y \) and \( X \neq Y \), \( X \) is a proper subset of \( Y \) (\( X \subset Y \)).
- Proper superset is similarly defined (denoted \( X \supset Y \)).
- The subset relation has the following three basic properties:
  - \( X \subseteq X \) (reflexive)
  - \( X \subseteq Y \) and \( Y \subseteq X \) implies that \( X = Y \) (antisymmetric)
  - \( X \subseteq Y \) and \( Y \subseteq Z \), then \( X \subseteq Z \) (transitive)

Empty Set and Power Set

- The empty set (denoted \( \emptyset \)) includes no elements.
- For any set \( X \), the empty set is a subset of it (i.e., \( \emptyset \subseteq X \)).
- Each set \( X \neq \emptyset \) has at least two subsets \( X \) and \( \emptyset \).
- Each \( x \in X \) is also a subset of \( X \) (i.e., \( \{ x \} \subseteq X \)).
- Similarly, each pair of objects makes up a subset.
- The power set of a set \( X \) is all subsets of \( X \) (denoted \( 2^X \)).
- The number of members of a set \( X \) is denoted \( |X| \).
- The number of members of \( 2^X \) is equal to \( 2^{|X|} \).
The union of two sets $X$ and $Y$ (denoted $X \cup Y$) is the set composed of all objects that are a member of either $X$ or $Y$ (i.e., $X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$).

The intersection of two sets $X$ and $Y$ (denoted $X \cap Y$) is set composed of all objects that are a member of both $X$ and $Y$ (i.e., $X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$).

Two sets $X$ and $Y$ are disjoint if their intersection contains no members (i.e., $X \cap Y = \emptyset$).

Otherwise, the sets intersect (i.e., $X \cap Y \neq \emptyset$).

A disjoint collection is a set of sets in which each pair of member sets is disjoint.

A partition of $X$ is a disjoint collection $\pi$ of nonempty subsets of $X$ such that each member of $X$ is contained within set in $\pi$.

Complements

The set $U$ is called the universal set.

The absolute complement of a set $X$ (denoted $\overline{X}$) are those elements in $U$ which are not in $X$ (i.e., $\{x \in U \mid x \notin X\}$).

The relative complement of a set $X$ with respect to a set $Y$ (denoted $Y - X$) are those elements in $Y$ which are not in $X$ (i.e., $Y - X = Y \cap \overline{X} = \{x \in Y \mid x \notin X\}$).

The symmetric difference of two sets $X$ and $Y$ (denoted $X + Y$) are those objects in exactly one of the two sets [i.e., $(A - B) \cup (B - A)$].

If $U = \{1,2,3,5,6,10,15,30\}$, $X = \{2,3\}$, and $Y = \{2,5\}$, then

$\overline{X} = X - Y = X + Y = \{1,5,10,15,30\}$

Binary Relations

Binary relations show relationships between two items.

Examples include things like “$a$ is less than $b$”.

An ordered pair is a set of two objects which have an order.

An ordered pair of $x$ and $y$ is denoted by $(x,y)$ and is equivalent to the set $\{(x) , \{x,y\}\}$.

A binary relation is simply a set of ordered pairs.

We say that $x$ is $p$-related to $y$ (denoted $xpy$) when $p$ is a binary relation and $(x,y) \in p$.

The domain and range of $p$ are

$D_p = \{x \mid \exists y . (x,y) \in p\}$

$R_p = \{y \mid \exists x . (x,y) \in p\}$

Ternary and $n$-ary Relations

An ordered triple $(x,y,z)$ is equivalent to the ordered pair $(\langle x,y \rangle, z)$.

A ternary relation is simply a set of ordered triples.

We can further define for any size $n$ an ordered $n$-tuple and use them to define $n$-ary relations.
Asynchronous Circuit Design

Example

- A binary relation \( p \) that says that \( x \) times \( y \) equals 30 is defined as follows:
  \[
  p = \{(1,30), (2,15), (3,10), (5,6), (6,5), (10,3), (15,2), (30,1)\}
  \]
- The cartesian product is the set of all pairs \((x, y)\), where \( x \) is a member of some set \( X \) and \( y \) is a member of some set \( Y \):
  \[
  X \times Y = \{(x,y) | x \in X \land y \in Y\}
  \]
- If \( X \supseteq Q_y \) and \( Y \supseteq R_y \), then \( p \subseteq X \times Y \) and \( p \) is a relation from \( X \) to \( Y \).
- The cartesian product of \( X = \{2,3,5\} \) and \( Y = \{6,10\} \) is defined as follows:
  \[
  X \times Y = \{(2,6), (2,10), (3,6), (3,10), (5,6), (5,10)\}
  \]

Equivalence Relations

- A relation \( p \) in a set \( X \) is an equivalence relation iff it is reflexive (i.e., \( x \rho x \) for all \( x \in X \)), symmetric (i.e., \( x \rho y \) implies \( y \rho x \)), and transitive (i.e., \( x \rho y \) and \( y \rho z \) imply \( x \rho z \)).
- A set \( A \subseteq X \) is an equivalence class iff there exists an \( x \in A \) such that \( A \) is equal to the set of all \( y \) for which \( x \rho y \).
- The equivalence class implied by \( x \) is denoted \([x]\).
- Using \( p \), we can partition a set \( X \) into a set of equivalence classes called a quotient set, which is denoted by \( X/p \).

Example

- The binary relation \( p \) on the set \( X = \{1,2,3,5,6,10,15,30\} \) defined below is an equivalence relation.
  \[
  p = \{(1,1), (2,2), (2,3), (2,5), (3,2), (3,3), (3,5), (5,2), (5,5), (6,6), (6,10), (6,15), (10,6), (10,10), (10,15), (15,6), (15,10), (15,15), (30,30)\}
  \]
- \( X/p = \{\{1\}, \{2\}, \{3\}, \{5\}, \{6\}, \{10\}, \{15\}, \{30\}\} \)

Functions

- A function is a binary relation in which no two members have the same first element.
- More formally, a binary relation \( f \) is a function if \((x, y)\) and \((x, z)\) are members of \( f \), then \( y = z \).
- If \( f \) is a function and \((x, y) \in f\) (i.e., \( x \rho y \)), then \( x \) is an argument of \( f \) and \( y \) is the image of \( x \) under \( f \).
- A function \( f \) is into \( Y \) if \( R_f \subseteq Y \).
- A function \( f \) is onto \( Y \) if \( R_f = Y \).
- A function \( f \) is one-to-one if \( f(x) = f(y) \) implies that \( x = y \).
- Functions can be extended to more variables by using arguments that are ordered \( n \)-tuples.

Partial Order

- A relation \( p \) is a partial order if it is reflexive, antisymmetric (i.e., \( x \rho y \) and \( y \rho x \) implies that \( x = y \)), and transitive.
- A partially ordered set (poset) is a pair \((X, \leq)\), where \( \leq \) partially orders \( X \).
- A partial order is a simple (or linear) ordering if for every pair of elements from the domain \( x \) and \( y \) either \( x \rho y \) or \( y \rho x \).
- An example of a simple ordering is \( \leq \) on the real numbers.
- A simply ordered set is also called a chain.
- Posets \((X, \leq)\) and \((X', \leq')\) are isomorphic if there exists a one-to-one mapping between \( X \) and \( X' \) that preserves order.
Example Posets

Least and Greatest Members

A least member of $X$ with respect to $\leq$ is a $x$ in $X$ such that $x \leq y$ for all $y$ in $X$.

- A least member is unique.
- A minimal member is a $x$ in $X$ such that there does not exist a $y$ in $X$ such that $y < x$.
- A minimal member need not be unique.
- A greatest member is a $x$ in $X$ such that $y \leq x$ for all $y$ in $X$.
- A maximal member is a $x$ in $X$ such that there does not exist a $y$ in $X$ such that $y > x$.
- A poset $(X, \leq)$ is well-ordered when each nonempty subset of $X$ has a least member. Any well-ordered set must be a chain.

Upper and Lower Bounds

For a poset $(X, \leq)$ and $A \subseteq X$, an element $x \in X$ is an upper bound for $A$ if for all $a \in A$, $a \leq x$.

- It is a least upper bound for $A$ [denoted lub$(A)$] if $x$ is an upper bound and $x \leq y$ for all $y$ which are upper bounds of $A$.
- Similarly, an element $x \in X$ is a lower bound for $A$ if for all $a \in A$, $x \leq a$.
- It is a greatest lower bound for $A$ [denoted glb$(A)$] if $x$ is a lower bound and $y \leq x$ for all $y$ which are lower bounds of $A$.
- If $A$ has a least upper bound, it is unique, and similarly for the greatest lower bound.