Asynchronous Circuit Design

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Lecture B: Sets and Relations
Appendix B

Sets

- A *set* S is any collection of objects.
- Each x in S is a *member* of S (denoted $x \in S$).
- When x is not a member of S, it is denoted by $x \notin S$.
- Two sets X and Y are equal when they consist of the same members (denoted $X = Y$).
- This means that if $X = Y$ and $a \in X$, then $a \in Y$.
- This is known as the *principle of extension*.
- If two sets are not equal, it is denoted $X \neq Y$.
- There are three basic properties of equality:
  1. $X = X$ (*reflexive*)
  2. $X = Y$ implies $Y = X$ (*symmetric*)
  3. $X = Y$ and $Y = Z$ then $X = Z$ (*transitive*)

Principle of Abstraction

- Large or infinite sets are described using the help of *predicates*.
- A predicate $P(x)$ takes an object and returns true or false.
- When a set S is defined using a predicate $P(x)$, the set S contains those objects a such that $P(a)$ is true.
- This is known as the *principle of abstraction*.
- This is denoted using *set builder notation* as follows:

$$S = \{x \mid P(x)\}$$

- Read as "the set of all objects x such that $P(x)$ is true."
- The following sets can be used interchangeably:

$$\begin{aligned} \{x \mid x \in A \text{ and } P(x)\} &= \{x \in A \mid P(x)\} \\ \{y \mid y = f(x) \text{ and } P(x)\} &= \{f(x) \mid P(x)\} \end{aligned}$$

Examples

- $\{x \in \mathcal{N} \mid x \text{ divides } 30\} =$
- Which of the following sets are equal to it?
  - $\{30, 15, 10, 6, 5, 3, 2, 1\}$
  - $\{1, 2, 3, 4, 5, 6, 10, 15, 30\}$
  - $\{1, 1, 2, 3, 5, 5, 6, 10, 15, 30\}$
  - $\{1, 2, 3, 5, 5, 10, 15, 30\}$

Subset

- If X and Y are sets such that every member of X is also a member of Y, then X is a subset of Y (denoted $X \subseteq Y$).
- If every member of Y is a member of X, then X is a superset of Y (denoted $X \supseteq Y$).
- If $X \subseteq Y$ and $X \neq Y$, X is a proper subset of Y ($X \subset Y$).
- Proper superset is similarly defined (denoted $X \supset Y$).
- The subset relation has the following three basic properties:
  1. $X \subseteq X$ (*reflexive*)
  2. $X \subseteq Y$ and $Y \subseteq X$ implies that $X = Y$ (*antisymmetric*)
  3. $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$ (*transitive*)

Empty Set and Power Set

- The *empty set* (denoted $\emptyset$) includes no elements.
- For any set X, the empty set is a subset of it (i.e., $\emptyset \subseteq X$).
- Each set $X \neq \emptyset$ has at least two subsets X and $\emptyset$.
- Each $x \in X$ is also a subset of X (i.e., $\{x\} \subseteq X$).
- Similarly, each pair of objects makes up a subset.
- The *power set* of a set X is all subsets of X (denoted 2^X).
- The number of members of a set X is denoted $|X|$.
- The number of members of 2^X is equal to $2^{|X|}$.

Unions, Intersections, and Partitions

- The *union* of two sets X and Y (denoted $X \cup Y$) is the set composed of all objects that are a member of either X or Y (i.e., $X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$).
- The *intersection* of two sets X and Y (denoted $X \cap Y$) is set composed of all objects that are a member of both X and Y (i.e., $X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$).
- Two sets X and Y are *disjoint* if their intersection contains no members (i.e., $X \cap Y = \emptyset$).
- Otherwise, the sets *intersect* (i.e., $X \cap Y \neq \emptyset$).
- A *disjoint collection* is a set of sets in which each pair of member sets is disjoint.
- A *partition* of X is a disjoint collection π of nonempty subsets of X such that each member of X is contained within set in π.

Examples

- If $X = \{2,3,5\}$ and $Y = \{1,2,3,5,6,10,15,30\}$, then

$$\begin{array}{ccc} X & ? & Y \\ 2^X & = & \end{array}$$

- If $X = \{2,3\}$ and $Y = \{2,5\}$, then

$$\begin{aligned} X \cup Y &= \\ X \cap Y &= \end{aligned}$$

- Partition of $\{1,2,3,5,6,10,15,30\}$?
  - $\{\{1\},\{3,5\},\{6,10,15\},\{30\}\}$
  - $\{\{1\},\{2,3,5\},\{5,6,10,15\},\{30\}\}$
  - $\{\{1\},\{2,3,5\},\{6,10,15\},\{30\}\}$

Complements

- The set U is called the *universal set*.
- The *absolute complement* of a set X (denoted $\overline{X}$) are those elements in U which are not in X (i.e., $\{x \in U \mid x \notin X\}$).
- The *relative complement* of a set X with respect to a set Y (denoted $Y - X$) are those elements in Y which are not in X (i.e., $Y - X = Y \cap \overline{X} = \{x \in Y \mid x \notin X\}$).
- The *symmetric difference* of two sets X and Y (denoted $X + Y$) are those objects in exactly one of the two sets [i.e., $(A - B) \cup (B - A)$].
- If $U = \{1,2,3,5,6,10,15,30\}$, $X = \{2,3\}$, and $Y = \{2,5\}$, then

$$\begin{aligned} \overline{X} &= \\ X - Y &= \\ X + Y &= \end{aligned}$$

Identities

Law	Union	Intersection
Associative	$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$
Communitive	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive	$A \cup (B \cap C) =$ $(A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) =$ $(A \cap B) \cup (A \cap C)$
Identity	$A \cup \emptyset = A$	$A \cap U = A$
Inverse	$A \cup \overline{A} = U$	$A \cap \overline{A} = \emptyset$
Idempotent	$A \cup A = A$	$A \cap A = A$
Absorption	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
DeMorgan	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$

- The *principle of duality* allows translation of any theorem in terms of $\cup$, $\cap$, and complement to a dual theorem.

Binary Relations

- *Binary relations* show relationships between two items.
- Examples include things like "a is less than b".
- An *ordered pair* is a set of two objects which have an order.
- An ordered pair of x and y is denoted by $\langle x,y \rangle$ and is equivalent to the set $\{\{x\},\{x,y\}\}$.
- A binary relation is simply a set of ordered pairs.
- We say that x is ρ-*related to* y (denoted $x \rho y$) when ρ is a binary relation and $\langle x,y \rangle \in \rho$.
- The *domain* and *range* of ρ are

$$\begin{aligned} D_\rho &= \{x \mid \exists y \,.\, \langle x,y \rangle \in \rho\} \\ R_\rho &= \{y \mid \exists x \,.\, \langle x,y \rangle \in \rho\} \end{aligned}$$

Ternary and *n*-ary Relations

- An *ordered triple* $\langle x,y,z \rangle$ is equivalent to the ordered pair $\langle \langle x,y \rangle, z \rangle$.
- A *ternary relation* is simply a set of ordered triples.
- We can further define for any size n an *ordered n-tuple* and use them to define *n-ary relations*.

Example

- A binary relation ρ that says that x times y equals 30 is defined as follows:

$$\rho = \{\langle 1,30\rangle, \langle 2,15\rangle, \langle 3,10\rangle, \langle 5,6\rangle, \langle 6,5\rangle, \langle 10,3\rangle, \langle 15,2\rangle, \langle 30,1\rangle\}$$

- The *cartesian product* is the set of all pairs $\langle x,y\rangle$, where x is a member of some set X and y is a member of some set Y.

$$X \times Y = \{\langle x,y\rangle \mid x \in X \wedge y \in Y\}$$

- If $X \supseteq D_\rho$ and $Y \supseteq R_\rho$, then $\rho \subseteq X \times Y$ and ρ is a *relation from X to Y*.
- The cartesian product of $X = \{2,3,5\}$ and $Y = \{6,10\}$ is defined as follows:

$$X \times Y =$$

Equivalence Relations

- A relation ρ in a set X is an *equivalence relation* iff it is reflexive (i.e., $x\rho x$ for all $x \in X$), symmetric (i.e., $x\rho y$ implies $y\rho x$), and transitive (i.e., $x\rho y$ and $y\rho z$ imply $x\rho z$).
- A set $A \subseteq X$ is an *equivalence class* iff there exists an $x \in A$ such that A is equal to the set of all y for which $x\rho y$.
- The equivalence class implied by x is denoted $[x]$.
- Using ρ, we can partition a set X into a set of equivalence classes called a *quotient set*, which is denoted by X/ρ.

Example

- The binary relation ρ on the set $X = \{1,2,3,5,6,10,15,30\}$ defined below is an equivalence relation.

$$\begin{aligned}\rho &= \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 2,5\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 3,5\rangle, \langle 5,2\rangle, \\ &\quad \langle 5,5\rangle, \langle 6,6\rangle, \langle 6,10\rangle, \langle 6,15\rangle, \langle 10,6\rangle, \langle 10,10\rangle, \\ &\quad \langle 10,15\rangle, \langle 15,6\rangle, \langle 15,10\rangle, \langle 15,15\rangle, \langle 30,30\rangle\} \\ X/\rho &=\end{aligned}$$

Functions

- A *function* is a binary relation in which no two members have the same first element.
- More formally, a binary relation f is a function if $\langle x,y\rangle$ and $\langle x,z\rangle$ are members of f, then $y = z$.
- If f is a function and $\langle x,y\rangle \in f$ (i.e., xfy), then x is an argument of f and y is the *image* of x under f.
- A function f is *into* Y if $R_f \subseteq Y$.
- A function f is *onto* Y if $R_f = Y$.
- A function f is *one-to-one* if $f(x) = f(y)$ implies that $x = y$.
- Functions can be extended to more variables by using arguments that are ordered n-tuples.

Example

- The function f on the set $X = \{1,2,3,5,6,10,15,30\}$ is defined as the result of dividing 30 by x.

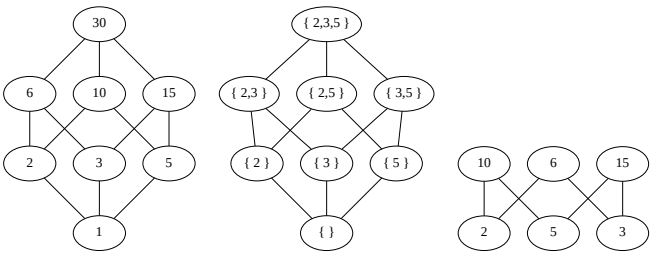
$$f = \{\langle 1,30\rangle, \langle 2,15\rangle, \langle 3,10\rangle, \langle 5,6\rangle, \langle 6,5\rangle, \langle 10,3\rangle, \langle 15,2\rangle, \langle 30,1\rangle\}$$

- Is it onto X?
- Is it one-to-one?

Partial Order

- A relation ρ is a *partial order* if it is reflexive, antisymmetric (i.e., $x\rho y$ and $y\rho x$ implies that $x = y$), and transitive.
- A *partially ordered set (poset)* is a pair $\langle X, \leq\rangle$, where $\leq$ partially orders X.
- A partial order is a *simple* (or *linear*) *ordering* if for every pair of elements from the domain x and y either $x\rho y$ or $y\rho x$.
- An example of a simple ordering is $\leq$ on the real numbers.
- A *simply ordered set* is also called a *chain*.
- Posets $\langle X, \leq\rangle$ and $\langle X', \leq'\rangle$ are isomorphic if there exists a one-to-one mapping between X and X' that preserves order.

Example Posets



Least and Greatest Members

- A *least member* of X with respect to $\leq$ is a x in X such that $x \leq y$ for all y in X.
- A least member is unique.
- A *minimal member* is a x in X such that there does not exist a y in X such that $y < x$.
- A minimal member need not be unique.
- A *greatest member* is a x in X such that $y \leq x$ for all y in X.
- A *maximal member* is a x in X such that there does not exist a y in X such that $y > x$.
- A poset $\langle X, \leq \rangle$ is *well-ordered* when each nonempty subset of X has a least member. Any well-ordered set must be a chain.

Upper and Lower Bounds

- For a poset $\langle X, \leq \rangle$ and $A \subseteq X$, an element $x \in X$ is an *upper bound* for A if for all $a \in A$, $a \leq x$.
- It is a *least upper bound* for A [denoted *lub*(A)] if x is an upper bound and $x \leq y$ for all y which are upper bounds of A.
- Similarly, an element $x \in X$ is a *lower bound* for A if for all $a \in A$, $x \leq a$.
- It is a *greatest lower bound* for A [denoted *glb*(A)] if x is a lower bound and $y \leq x$ for all y which are lower bounds of A.
- If A has a least upper bound, it is unique, and similarly for the greatest lower bound.