Asynchronous Circuit Design

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Lecture B: Sets and Relations
Appendix B
A set $S$ is any collection of objects.

Each $x$ in $S$ is a member of $S$ (denoted $x \in S$).

When $x$ is not a member of $S$, it is denoted by $x \notin S$.

Two sets $X$ and $Y$ are equal when they consist of the same members (denoted $X = Y$).

This means that if $X = Y$ then $a \in X$ implies $a \in Y$ and vice versa.

This is known as the principle of extension.

If two sets are not equal, it is denoted $X \neq Y$.

There are three basic properties of equality:

1. $X = X$ (reflexive)
2. $X = Y$ implies $Y = X$ (symmetric)
3. $X = Y$ and $Y = Z$ then $X = Z$ (transitive)
Principle of Abstraction

- Large or infinite sets are described using the help of *predicates*.
- A predicate $P(x)$ takes an object and returns true or false.
- When a set $S$ is defined using a predicate $P(x)$, the set $S$ contains those objects $a$ such that $P(a)$ is true.
- This is known as the *principle of abstraction*.
- This is denoted using *set builder notation* as follows:

$$S = \{ x \mid P(x) \}$$

- Read as “the set of all objects $x$ such that $P(x)$ is true.”
- The following sets can be used interchangeably:

$$\{ x \mid x \in A \text{ and } P(x) \} = \{ x \in A \mid P(x) \}$$
$$\{ y \mid y = f(x) \text{ and } P(x) \} = \{ f(x) \mid P(x) \}$$
Examples

\[ \{ x \in \mathbb{N} \mid x \text{ divides } 30 \} = \]

Which of the following sets are equal to it?

- \{30, 15, 10, 6, 5, 3, 2, 1\}
- \{1, 2, 3, 4, 5, 6, 10, 15, 30\}
- \{1, 1, 2, 3, 5, 5, 6, 10, 15, 30\}
- \{1, 2, 3, 5, 5, 10, 15, 30\}
If \( X \) and \( Y \) are sets such that every member of \( X \) is also a member of \( Y \), then \( X \) is a subset of \( Y \) (denoted \( X \subseteq Y \)).

If every member of \( Y \) is a member of \( X \), then \( X \) is a superset of \( Y \) (denoted \( X \supseteq Y \)).

If \( X \subseteq Y \) and \( X \neq Y \), \( X \) is a proper subset of \( Y \) (\( X \subset Y \)).

Proper superset is similarly defined (denoted \( X \supset Y \)).

The subset relation has the following three basic properties:

1. \( X \subseteq X \) (reflexive)
2. \( X \subseteq Y \) and \( Y \subseteq X \) implies that \( X = Y \) (antisymmetric)
3. \( X \subseteq Y \) and \( Y \subseteq Z \), then \( X \subseteq Z \) (transitive)
The *empty set* (denoted $\emptyset$) includes no elements.

For any set $X$, the empty set is a subset of it (i.e., $\emptyset \subseteq X$).

Each set $X \neq \emptyset$ has at least two subsets $X$ and $\emptyset$.

For each $x \in X$ there is a corresponding subset of $X$ (i.e., $\{x\} \subseteq X$).

Similarly, each pair of objects makes up a subset.

The *power set* of a set $X$ is all subsets of $X$ (denoted $2^X$).

The number of members of a set $X$ is denoted $|X|$.

The number of members of $2^X$ is equal to $2^{|X|}$.
The union of two sets $X$ and $Y$ (denoted $X \cup Y$) is the set composed of all objects that are a member of either $X$ or $Y$ (i.e., $X \cup Y = \{ x \mid x \in X \text{ or } x \in Y \}$).

The intersection of two sets $X$ and $Y$ (denoted $X \cap Y$) is set composed of all objects that are a member of both $X$ and $Y$ (i.e., $X \cap Y = \{ x \mid x \in X \text{ and } x \in Y \}$).

Two sets $X$ and $Y$ are disjoint if their intersection contains no members (i.e., $X \cap Y = \emptyset$).

Otherwise, the sets intersect (i.e., $X \cap Y \neq \emptyset$).

A disjoint collection is a set of sets in which each pair of member sets is disjoint.

A partition of $X$ is a disjoint collection $\pi$ of nonempty subsets of $X$ such that each member of $X$ is contained within a set in $\pi$. 
Examples

- If \( X = \{2, 3, 5\} \) and \( Y = \{1, 2, 3, 5, 6, 10, 15, 30\} \), then
  \[
  X \cup Y = \]
  \[
  2^X = \]

- If \( X = \{2, 3\} \) and \( Y = \{2, 5\} \), then
  \[
  X \cup Y = \]
  \[
  X \cap Y = \]

- Partition of \( \{1, 2, 3, 5, 6, 10, 15, 30\} \)?
  - \( \{\{1\}, \{3, 5\}, \{6, 10, 15\}, \{30\}\} \)
  - \( \{\{1\}, \{2, 3, 5\}, \{5, 6, 10, 15\}, \{30\}\} \)
  - \( \{\{1\}, \{2, 3, 5\}, \{6, 10, 15\}, \{30\}\} \)
Complements

- The set $U$ is called the *universal set*.
- The *absolute complement* of a set $X$ (denoted $\overline{X}$) are those elements in $U$ which are not in $X$ (i.e., $\{x \in U \mid x \notin X\}$).
- The *relative complement* of a set $X$ with respect to a set $Y$ (denoted $Y - X$) are those elements in $Y$ which are not in $X$ (i.e., $Y - X = Y \cap \overline{X} = \{x \in Y \mid x \notin X\}$).
- The *symmetric difference* of two sets $X$ and $Y$ (denoted $X + Y$) are those objects in exactly one of the two sets [i.e., $(A - B) \cup (B - A)$].
- If $U = \{1, 2, 3, 5, 6, 10, 15, 30\}$, $X = \{2, 3\}$, and $Y = \{2, 5\}$, then
  
  $$\begin{align*}
  \overline{X} &= \quad \\
  X - Y &= \quad \\
  X + Y &=
  \end{align*}$$
### Identities

<table>
<thead>
<tr>
<th>Law</th>
<th>Union</th>
<th>Intersection</th>
</tr>
</thead>
<tbody>
<tr>
<td>Associative</td>
<td>$A \cup (B \cup C) = (A \cup B) \cup C$</td>
<td>$A \cap (B \cap C) = (A \cap B) \cap C$</td>
</tr>
<tr>
<td>Commutative</td>
<td>$A \cup B = B \cup A$</td>
<td>$A \cap B = B \cap A$</td>
</tr>
<tr>
<td>Distributive</td>
<td>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$</td>
<td>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$</td>
</tr>
<tr>
<td>Identity</td>
<td>$A \cup \emptyset = A$</td>
<td>$A \cap U = A$</td>
</tr>
<tr>
<td>Inverse</td>
<td>$A \cup \overline{A} = U$</td>
<td>$A \cap \overline{A} = \emptyset$</td>
</tr>
<tr>
<td>Idempotent</td>
<td>$A \cup A = A$</td>
<td>$A \cap A = A$</td>
</tr>
<tr>
<td>Absorption</td>
<td>$A \cup (A \cap B) = A$</td>
<td>$A \cap (A \cup B) = A$</td>
</tr>
<tr>
<td>DeMorgan</td>
<td>$\overline{A \cup B} = \overline{A} \cap \overline{B}$</td>
<td>$\overline{A \cap B} = \overline{A} \cup \overline{B}$</td>
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</tbody>
</table>

- The *principle of duality* allows translation of any theorem in terms of $\cup$, $\cap$, and complement to a dual theorem.
Binary relations show relationships between two items.
Examples include things like “a is less than b”.
An ordered pair is a set of two objects which have an order.
An ordered pair of \( x \) and \( y \) is denoted by \( \langle x, y \rangle \) and is equivalent to the set \( \{\{x\}, \{x, y\}\} \).
A binary relation is simply a set of ordered pairs.
We say that \( x \) is \( \rho \)-related to \( y \) (denoted \( x \rho y \)) when \( \rho \) is a binary relation and \( \langle x, y \rangle \in \rho \).
The domain and range of \( \rho \) are
\[
D_{\rho} = \{ x \mid \exists y . \langle x, y \rangle \in \rho \} \\
R_{\rho} = \{ y \mid \exists x . \langle x, y \rangle \in \rho \}
\]
Ternary and $n$-ary Relations

- An ordered triple $\langle x, y, z \rangle$ is equivalent to the ordered pair $\langle \langle x, y \rangle, z \rangle$.
- A ternary relation is simply a set of ordered triples.
- We can further define for any size $n$ an ordered $n$-tuple and use them to define $n$-ary relations.
Example

- A binary relation $\rho$ that says that $x$ times $y$ equals 30 is defined as follows:
  
  \[
  \rho = \{ \langle 1, 30 \rangle, \langle 2, 15 \rangle, \langle 3, 10 \rangle, \langle 5, 6 \rangle, \langle 6, 5 \rangle, \langle 10, 3 \rangle, \langle 15, 2 \rangle, \langle 30, 1 \rangle \}
  \]

- The *cartesian product* is the set of all pairs $\langle x, y \rangle$, where $x$ is a member of some set $X$ and $y$ is a member of some set $Y$.

  \[
  X \times Y = \{ \langle x, y \rangle \mid x \in X \land y \in Y \}
  \]

- If $X \supseteq D_\rho$ and $Y \supseteq R_\rho$, then $\rho \subseteq X \times Y$ and $\rho$ is a *relation from $X$ to $Y$*.

- The cartesian product of $X = \{2, 3, 5\}$ and $Y = \{6, 10\}$ is defined as follows:

  \[
  X \times Y =
  \]
A relation $\rho$ in a set $X$ is an *equivalence relation* iff it is reflexive (i.e., $x \rho x$ for all $x \in X$), symmetric (i.e., $x \rho y$ implies $y \rho x$), and transitive (i.e., $x \rho y$ and $y \rho z$ imply $x \rho z$).

A set $A \subseteq X$ is an *equivalence class* iff there exists an $x \in A$ such that $A$ is equal to the set of all $y$ for which $x \rho y$.

The equivalence class implied by $x$ is denoted $[x]$.

Using $\rho$, we can partition a set $X$ into a set of equivalence classes called a *quotient set*, which is denoted by $X/\rho$. 

The binary relation \( \rho \) on the set \( X = \{1, 2, 3, 5, 6, 10, 15, 30\} \) defined below is an equivalence relation.

\[
\rho = \{ (1, 1), (2, 2), (2, 3), (2, 5), (3, 2), (3, 3), (3, 5), (5, 2), \\
(5, 3), (5, 5), (6, 6), (6, 10), (6, 15), (10, 6), \\
(10, 10), (10, 15), (15, 6), (15, 10), (15, 15), (30, 30) \}
\]

\[
X / \rho =
\]
A function is a binary relation in which no two members have the same first element.

More formally, a binary relation \( f \) is a function if \( \langle x, y \rangle \) and \( \langle x, z \rangle \) are members of \( f \), then \( y = z \).

If \( f \) is a function and \( \langle x, y \rangle \in f \) (i.e., \( xfy \)), then \( x \) is an argument of \( f \) and \( y \) is the image of \( x \) under \( f \).

A function \( f \) is into \( Y \) if \( R_f \subseteq Y \).

A function \( f \) is onto \( Y \) if \( R_f = Y \).

A function \( f \) is one-to-one if \( f(x) = f(y) \) implies that \( x = y \).

Functions can be extended to more variables by using arguments that are ordered \( n \)-tuples.
The function $f$ on the set $X = \{1, 2, 3, 5, 6, 10, 15, 30\}$ is defined as the result of dividing 30 by $x$.

$$f = \{\langle 1, 30 \rangle, \langle 2, 15 \rangle, \langle 3, 10 \rangle, \langle 5, 6 \rangle, \langle 6, 5 \rangle, \langle 10, 3 \rangle, \langle 15, 2 \rangle, \langle 30, 1 \rangle\}$$

Is it onto $X$?
Is it one-to-one?
A relation $\rho$ is a *partial order* if it is reflexive, antisymmetric (i.e., $x \rho y$ and $y \rho x$ implies that $x = y$), and transitive.

A *partially ordered set (poset)* is a pair $\langle X, \leq \rangle$, where $\leq$ partially orders $X$.

A partial order is a *simple (or linear) ordering* if for every pair of elements from the domain $x$ and $y$ either $x \rho y$ or $y \rho x$.

An example of a simple ordering is $\leq$ on the real numbers.

A *simply ordered set* is also called a *chain*.

Posets $\langle X, \leq \rangle$ and $\langle X', \leq' \rangle$ are isomorphic if there exists a one-to-one mapping between $X$ and $X'$ that preserves order.
Example Posets

- Poset 1:
  - Elements: $\{1, 2, 3, 5, 6, 10, 15, 30\}$
  - Partial order:
    - $1 \leq 2, 3, 5$
    - $2 \leq 6$
    - $3 \leq 5, 6, 10$
    - $5 \leq 10, 15$
    - $6 \leq 15$
    - $10 \leq 15$
    - $15 \leq 30$
  - Sets:
    - $\{1\}$
    - $\{2\}$
    - $\{3\}$
    - $\{5\}$
    - $\{2, 3\}$
    - $\{2, 5\}$
    - $\{3, 5\}$
    - $\{2, 3, 5\}$

- Poset 2:
  - Elements: $\{1, 2, 3, 5, 6, 10, 15\}$
  - Partial order:
    - $1 \leq 2, 3, 5$
    - $2 \leq 6$
    - $3 \leq 5, 6, 10$
    - $5 \leq 10, 15$
    - $6 \leq 15$
    - $10 \leq 15$
  - Sets:
    - $\{1\}$
    - $\{2\}$
    - $\{3\}$
    - $\{5\}$
    - $\{2, 3\}$
    - $\{2, 5\}$
    - $\{3, 5\}$
    - $\{2, 3, 5\}$

- Poset 3:
  - Elements: $\{2, 3, 5, 6, 10, 15\}$
  - Partial order:
    - $2 \leq 6$
    - $3 \leq 5, 6, 10$
    - $5 \leq 10, 15$
    - $6 \leq 15$
    - $10 \leq 15$
  - Sets:
    - $\{2\}$
    - $\{3\}$
    - $\{5\}$
    - $\{2, 3\}$
    - $\{2, 5\}$
    - $\{3, 5\}$
    - $\{2, 3, 5\}$

- Poset 4:
  - Elements: $\{2, 3, 5, 6, 10, 15\}$
  - Partial order:
    - $2 \leq 6$
    - $3 \leq 5, 6, 10$
    - $5 \leq 10, 15$
    - $6 \leq 15$
    - $10 \leq 15$
  - Sets:
    - $\{2\}$
    - $\{3\}$
    - $\{5\}$
    - $\{2, 3\}$
    - $\{2, 5\}$
    - $\{3, 5\}$
    - $\{2, 3, 5\}$
A least member of $X$ with respect to $\leq$ is a $x$ in $X$ such that $x \leq y$ for all $y$ in $X$.

A least member is unique.

A minimal member is a $x$ in $X$ such that there does not exist a $y$ in $X$ such that $y < x$.

A minimal member need not be unique.

A greatest member is a $x$ in $X$ such that $y \leq x$ for all $y$ in $X$.

A maximal member is a $x$ in $X$ such that there does not exist a $y$ in $X$ such that $y > x$.

A poset $\langle X, \leq \rangle$ is well-ordered when each nonempty subset of $X$ has a least member. Any well-ordered set must be a chain.
For a poset \( \langle X, \leq \rangle \) and \( A \subseteq X \), an element \( x \in X \) is an upper bound for \( A \) if for all \( a \in A \), \( a \leq x \).

It is a least upper bound for \( A \) [denoted \( \text{lub}(A) \)] if \( x \) is an upper bound and \( x \leq y \) for all \( y \) which are upper bounds of \( A \).

Similarly, an element \( x \in X \) is a lower bound for \( A \) if for all \( a \in A \), \( x \leq a \).

It is a greatest lower bound for \( A \) [denoted \( \text{glb}(A) \)] if \( x \) is a lower bound and \( y \leq x \) for all \( y \) which are lower bounds of \( A \).

If \( A \) has a least upper bound, it is unique, and similarly for the greatest lower bound.